ED-2811

M.A./M.Sc. (Final) Examination, 2021

MATHEMATICS

Optional

Paper - IV (ii)

Wavelets

Time : Three Hours] [*Maximum Marks* : 100

Note : Answer any **two** parts from each question. All questions carry equal marks.

Unit-I

1. (*a*) State and prove Balian-low theorem for $g \in L^2(\mathbb{R})$.

(*b*) Let $\{V_j : j \in \mathbb{Z}\}$ be a sequence of closed subspaces of $L^2(\mathbb{R})$ satisfying

(*i*) $V_j \subseteq V_{j+1} \quad \forall\ j \in \mathbb{Z}$

(ii) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ $\forall\, j \in \mathbb{Z}$

(iii) There exists a function $\phi \in V_0$, such that $\{\phi(x-k) \mid k \in \mathbb{Z}\}$

is an orthonormal basis for V_0 and $|\hat{\phi}|$ is continuous at 0. Then prove that the following two conditions are equivalent :

(i) $\hat{\phi}(0) = 0$,

(ii) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$

(c) Let r be a non-negative integer. Let ψ be a function in $C^r(\mathbb{R})$ such that

$|\psi(x)| \leq \dfrac{C}{(1+|x|)^{r+1+\epsilon}}$ for some $\epsilon > 0$,

and that $\psi^{(m)} \in L^\infty(\mathbb{R})$ for $m = 1, 2, \ldots r$. If $\{\psi_{j,k} \mid j, k \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$, then all moments of ψ upto order r are zero; i.e.

$\int_{\mathbb{R}} x^m \psi(x)\, dx = 0 \;\forall\; m = 0, 1, 2 \ldots r.$

Unit-II

2. (*a*) Suppose $f \in L^2(\mathbb{R})$, then prove that f is orthogonal to W_j if and only if $\sum_{k\in\mathbb{Z}} \hat{f}\left(\xi + 2^{j+1}k\pi\right)\overline{\hat{\psi}\left(2^{-j}\xi + 2k\pi\right)} = 0$ for a.e. $\xi \in \mathbb{R}$.

(*b*) Suppose $\psi \in L^2(\mathbb{R})$ and $b = |\hat{\psi}|$ has support contained in $\left[\frac{-8}{3}\pi, \frac{-2}{3}\pi\right] \bigcup \left[\frac{2}{3}\pi, \frac{8}{3}\pi\right]$. Then prove that ψ is an orthonormal wavelet if and only if

(*i*) $b^2(\xi) + b^2\left(\frac{\xi}{2}\right) = 1$ for $\xi \in \left[\frac{4}{3}\pi, \frac{8}{3}\pi\right]$

(*ii*) $b^2(\xi) + b^2(\xi + 2\pi) = 1$ for $\xi \in \left[\frac{-4}{3}\pi, \frac{-2}{3}\pi\right]$

(*iii*) $b(\xi) = b\left(\frac{\xi}{2} + 2\pi\right)$ for $\xi \in \left[\frac{-8}{3}\pi, \frac{-4}{3}\pi\right]$

(c) Suppose $N = 1, 2, 3, ...,$ then prove that

$$\sum_{k\in\mathbb{Z}} \frac{1}{(\xi + 2k\pi)^{N+1}} = \left(2\sin\left(\frac{1}{2}\xi\right)\right)^{-N-1} P_N\left(\frac{\xi}{2}\right),$$

where P_N is a trigonometric polynomial satisfying :

(i) P_N is an even function, and $P_N(k\pi) = (-1)^{k(N-1)}\ \forall\ k \in \mathbb{Z}$.

(ii) when N is odd, P_N is π-periodic and $P_N(\xi) > 0$ for all $\xi \in \mathbb{R}$.

(iii) when N is even, $P_N(\xi + \pi) = -P_N(\xi)$ for all $\xi \in \mathbb{R}$.

Unit-III

3. (a) Let H be a Hilbert space and $\{e_j : j = 1, 2, ...\}$ be a family of elements of H, then prove that

(i) $\| f \|^2 = \sum_{j=1}^{\infty} |< f, e_j >|^2$ holds for all $f \in H$ if and only if (*ii*) $f = \sum_{j=1}^{\infty} < f, e_j > e_j$ with convergence in H, for all $f \in H$.

(*b*) Prove that

$$\sum_{j \in D} 2^{-j} \int_{\mathbb{R}} | \hat{f}\left(2^{-j}\xi\right) \hat{\psi}(\xi) | \sum_{k \neq 0} | \hat{f}\left(2^{-j}(\xi + 2k\pi)\right) \hat{\psi}(\xi + 2k\pi) | \, d\xi < \infty$$

(*c*) If ψ is an orthonormal wavelet, then prove that

$$\hat{\psi}\left(2^n \xi\right) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \hat{\psi}\left(2^n(\xi + 2k\pi)\right) \overline{\hat{\psi}\left(2^j(\xi + 2k\pi)\right)} \hat{\psi}\left(2^j(\xi)\right)$$

for all $n \geq 1$.

Unit-IV

4. (*a*) Suppose that $g \in L^2(\mathbb{R})$ and that $\left\{g_{m,n}(x) = e^{2\pi imx} g(x-n) \mid m, n \in \mathbb{Z}\right\}$ is a frame for $L^2(\mathbb{R})$. Let $\hat{g} \equiv S^{-1}\dot{g}$, where $S = F^* F$ and F is a frame operator. Then prove that

$$\widetilde{g_{m,n}}(x) = e^{2\pi imx} \tilde{g}(x-n) = \tilde{g}_{m,n(x)}$$

for $m, n \in \mathbb{Z}$.

(*b*) Let $\psi \in L^2(\mathbb{R})$ be such that

$$A_\psi = \underline{S}_\psi - \sum_{q \in 2\mathbb{Z}+1} \left[\beta_\psi(q)\beta_\psi(-q)\right]^{\frac{1}{2}} > 0 \text{ and}$$

$$B_\psi = \overline{S}_\psi + \sum_{q \in 2\mathbb{Z}+1} \left[\beta_\psi(q)\beta_\psi(-q)\right]^{\frac{1}{2}} < \infty .$$

Then prove that $\{\psi_{j,k} \mid j,k \in \mathbb{Z}\}$ is a frame with frame bounds A_ψ and B_ψ.

(*c*) Suppose that $g \in L^2(\mathbb{R})$ and that $\left\{g_{m,n}(x) = e^{2\pi imx} g(x-n) \mid m,n \in \mathbb{Z}\right\}$ is a frame with frame bounds A and B. Then prove that $0 < A \leq |Rg(s,t)|^2 \leq B < \infty$ and

$$\left(R\tilde{g}_{m,n}\right)(s,t) = \frac{e^{2\pi ims} e^{2\pi int}}{Rg(s,t)} .$$

Unit-V

5. (*a*) If $N = 2^q$ then prove that $C_N = E_1, E_2 \ldots E_q$, where each E_j is an $N \times N$ matrix such that each row has precisely two non-zero entries.

(*b*) Prove that the sequence $\left\{u_{j,k} \mid j \in \mathbb{Z}, 0 \le k \le l_j - 1\right\}$ given by

$$u_{j,k}(x) = \sqrt{\frac{2}{l_j}}\, w_j(x) \cos\left(\pi\left(k + \frac{1}{2}\right)\left(\frac{x - a_j}{l_j}\right)\right) x \in \mathbb{Z}$$

is an orthonormal basis for $l^2(\mathbb{Z})$.

(*c*) What do you mean by decomposition of wavelets ? Write in detail "how Haar wavelet works for doing the decomposition algorithm".

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